

The method of vacuum vectors in the theory of Yang – Baxter equation

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Translated from the book: Applied Problems in Calculus, Publishing House of Chelyabinsk Polytechnical Institute, Chelyabinsk, Russia, 1986, pages 39–48

Abstract

In modern terminology, this is the first published paper where the solutions of Yang–Baxter equation “at roots of unity” were analyzed and shown to be related to algebraic curves of genus > 1 . They are also known now to be connected with the “chiral Potts model”. The paper’s abstract as written in 1986 reads: “Vacuum vectors of an L -operator form a holomorphic bundle over the vacuum curve of that operator. These notions, as well as the theory of commutation relations of the 6-vertex model, are used in this work for constructing solutions of the Yang–Baxter equation that do not possess a spectral parameter of traditional type”.

In this paper the properties of solutions of Yang–Baxter equation are studied connected with the existence of the so-called vacuum vectors of those solutions. In §1 general definitions and theorems are stated concerning vacuum vectors and their relation to Yang–Baxter equation. In §2 and 3 the problem of existence of some new solutions to Yang–Baxter equation is solved on the base of those definitions and theorems. These solutions are intimately connected with the well-known “6-vertex model” and at the same time possess remarkable new properties considered in §4.

§1. Vacuum vectors and vacuum curves

Let S^Q and S^A be two finite-dimensional complex linear spaces (S^Q is called “quantum space”, and S^A is “auxiliary”, see [1]); and let L be a linear operator acting in $S^Q \otimes S^A$. We will denote vectors from S^Q by letters U, V, \dots , and vectors from S^A by letters X, Y, \dots . Let the relation

$$L(U \otimes X) = V \otimes Y \tag{1}$$

hold.

Definition 1. The *vacuum variety* Γ_L of an operator L is the set of pairs (U, V) , $U \neq 0$, $V \neq 0$, for which the relation (1) is satisfied with some X and Y , $X \neq 0$, taken to within the equivalence $(U, V) \sim (t_1 U, t_2 V)$, where t_1 and t_2 are some nonzero numbers. Any (including zero) vector X corresponding according to (1) to a given point $z \in \Gamma_L$ is called *vacuum vector*,

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while any vector Y — *co-vacuum vector*. The vacuum variety is an algebraic variety, and a point z must be assumed to enter it with the multiplicity equal to the dimension of the space of vacuum vectors in it.

Definition 2. An operator L acting in $S^Q \otimes S^A$ is called *equivalent* to an operator M acting in $S^Q \otimes S^B$ if an isomorphism $R: S^A \rightarrow S^B$ exists such that $RL = MR$. Similarly, two families of operators $L(\lambda)$ and $M(\lambda)$ (λ is any parameter) are called equivalent if an isomorphism R exists such that $RL(\lambda) = M(\lambda)R$ for every λ . In these cases we will say that R performs such equivalence.

Remark. Here we identify the operator R with the operator $\mathbf{1} \otimes R$ acting from $S^Q \otimes S^A$ to $S^Q \otimes S^B$. We also treat similarly other operators when needed.

The following lemma, although being evident, plays the fundamental role in the sequel.

Lemma 1. The vacuum varieties of equivalent operators coincide.

In the rest of this paper we consider only the case $\dim S^Q = 2$. We can then put in formula (1) $U = \begin{pmatrix} u \\ 1 \end{pmatrix}$, $V = \begin{pmatrix} v \\ 1 \end{pmatrix}$, where u and v are complex numbers or ∞ . Denote $\dim S^A$ as N .

Theorem 1 [2]. The vacuum curve Γ_L of a generic operator L is a smooth irreducible algebraic curve given by equation of the kind

$$P_L(u, v) = \sum_{i,j=1}^N a_{ij} u^i v^j = 0. \quad (2)$$

The vacuum (co-vacuum) vectors form a one-dimensional holomorphic bundle of degree $N^2 - N$ over Γ_L .

Degenerate cases are also important, for example the case when $P_L(u, v) = [p(u, v)]^l$. In such case the vacuum (co-vacuum) vectors form an l -dimensional holomorphic bundle over the curve given by the polynomial $p(u, v)$ [2].

Introduce now two auxiliary spaces S_1^A and S_2^A , and let $S^A = S_1^A \otimes S_2^A$. Consider operators L_1 and L_2 acting respectively in $S^Q \otimes S_1^A$ and $S^Q \otimes S_2^A$, and the operator $L_1 L_2$ acting in $S^Q \otimes S^A$.

Definition 3. *Composition* of vacuum curves Γ_1 and Γ_2 given by equations $P_1(u, v) = 0$ and $P_2(u, v) = 0$ of the form (2) is the curve Γ given by equation $P(u, v) = 0$ where $P(u, v)$ is the resultant [3] of $P_1(w, v)$ and $P_2(u, w)$ considered as polynomials in w .

Thus, Γ is the set of points u, v , taken with proper multiplicities, for which a w exists such that $(u, w) \in \Gamma_2$ and $(w, v) \in \Gamma_1$.

Theorem 2 [2]. The vacuum curve Γ of the operator $L_1 L_2$ is the composition of vacuum curves Γ_1 and Γ_2 of operators L_1 and L_2 .

We are interested in the equivalence of operators of the form $L_1 L_2$ and $L_2 L_1$, that is in solutions of Yang–Baxter equation $RL_1 L_2 = L_2 L_1 R$. This motivates, in light of Lemma 1 and Theorem 2, the following

Definition 4. Vacuum curves Γ_1 and Γ_2 are called *commuting* if the composition of Γ_1 and Γ_2 coincides with the composition of Γ_2 and Γ_1 .

§2. Formulation of the main theorem

Recall that $\dim S^Q = 2$, and let $\dim S_1^Q = \dim S_2^Q = 3$. Considers two one-parametric families of operators:

$$L_i(\lambda) = \begin{pmatrix} A_i(\lambda) & B_i(\lambda) \\ C_i(\lambda) & D_i(\lambda) \end{pmatrix} \quad (3)$$

($i = 1, 2$), where $A_i(\lambda), \dots, D_i(\lambda)$ act in spaces S_i^A , realize two representations of the commutation relations of 6-vertex model [1,4–7] with parameter $\eta = \pi/3$, and are expressed, in the proper bases, by formulas

$$A_i(\lambda) = a_i \begin{pmatrix} \sin(\lambda + \rho_i) & & \\ & \sin(\lambda + \rho_i - \frac{2\pi}{3}) & \\ & & \sin(\lambda + \rho_i - \frac{4\pi}{3}) \end{pmatrix}, \quad (4)$$

$$D_i(\lambda) = d_i \begin{pmatrix} \sin(\lambda + \sigma_i - \frac{4\pi}{3}) & & \\ & \sin(\lambda + \sigma_i - \frac{2\pi}{3}) & \\ & & \sin(\lambda + \sigma_i) \end{pmatrix}, \quad (5),$$

$$B_i(\lambda) = C_i^T(\lambda) = \begin{pmatrix} 0 & & b_{13}^{(i)} \\ b_{21}^{(i)} & 0 & \\ & b_{32}^{(i)} & 0 \end{pmatrix}, \quad (6)$$

$$(b_{k+1,k}^{(i)})^2 - (b_{k,k-1}^{(i)})^2 = a_i d_i \sin \frac{2\pi}{3} \sin(\rho_i - \sigma_i - \frac{2+4k}{3}\pi). \quad (7)$$

Here ρ and σ are constant numbers; $k = 1, \dots, 3$, the addition in subscripts is understood modulo 3.

The method for calculation of vacuum curves is given in the work [2]. As a result of direct calculations we get for the vacuum curve of operator $L_i(\lambda)$ the equation

$$1 + \alpha_i(\lambda)u^3 - \delta_i(\lambda)v^3 - u^3v^3 = 0, \quad (8)$$

where

$$\alpha_i(\lambda) = -\frac{a_i^3 \sin 3(\lambda + \rho_i)}{4b_{21}^{(i)}b_{32}^{(i)}b_{13}^{(i)}}; \quad \delta_i(\lambda) = -\frac{d_i^3 \sin 3(\lambda + \sigma_i)}{4b_{21}^{(i)}b_{32}^{(i)}b_{13}^{(i)}}. \quad (9)$$

Lemma 2. Two vacuum curves given by equations

$$1 + \alpha u^3 - \delta v^3 - u^3v^3 = 0 \quad (10)$$

and

$$1 + \alpha' u^3 - \delta' v^3 - u^3v^3 = 0$$

commute if and only if $\alpha - \delta = \alpha' - \delta'$.

Proof is obtained by straightforward calculation.

Theorem 3. In order that two operator families $L(\lambda) = L_1(\lambda)L_2(\lambda)$ and $M(\lambda) = L_2(\lambda)L_1(\lambda)$, where $L_1(\lambda)$ and $L_2(\lambda)$ are given by formulas (3–7), be equivalent it is necessary and sufficient that the vacuum curves of operators $L_1(\lambda)$ and $L_2(\lambda)$ commute for every λ .

§3. Proof of Theorem 3

The necessity of the condition of Theorem 3 is obvious from Lemma 1. The rest of this section is devoted to the proof of sufficiency.

We will need the following notations. We will always understand the operator product of the type $\mathcal{L} = L_1 \dots L_k$ in the sense that those operators have a common quantum space S^Q and different auxiliary spaces S_1^A, \dots, S_k^A , so that \mathcal{L} acts in $S^Q \otimes S_1^A \otimes \dots \otimes S_k^A$. Similarly to (3), we will write \mathcal{L} as $\mathcal{L} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$. Next, if $L_i(\lambda)$ is an operator of the form (3–7) then we denote as $L_i^\dagger(\lambda)$ the operator of the same form given by the formula

$$L_i^\dagger(\lambda) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot L_i \lambda \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If $L_i(\lambda)$ and $L_i^\dagger(\lambda)$ enter in the same product of operators we will assume that they act in *different* copies of their auxiliary space.

Lemma 3. If the vacuum curve of operator $L_i(\lambda)$ has the equation (8) then the vacuum curve of operator $L_i^\dagger(\lambda)$ has the equation

$$1 - \delta_i(\lambda)u^3 + \alpha_i(\lambda)v^3 - u^3v^3 = 0.$$

The composition of vacuum curves of operators $L_i(\lambda)$ and $L_i^\dagger(\lambda)$ has the equation

$$(v^3 - u^3)^3 = 0. \tag{11}$$

Proof is performed by direct calculations.

Let us introduce the operation $\hat{}$ of “transposing in quantum indices”: if $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ then $\hat{L} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$.

Lemma 4. If an operator $L_i(\lambda)$ is given by formulas (3–7) then

$$\begin{aligned} \det L_i(\lambda) &= [a_i d_i \sin^2 \lambda - (b_{13}^{(i)})^2]^3, \\ \det \hat{L}_i(\lambda) &= \det L_i(\lambda + \frac{2\pi}{3}). \end{aligned}$$

Proof of the lemma is obtained by direct calculation.

Thus, all zeros of functions $\det L_i(\lambda)$ and $\det \hat{L}_i(\lambda)$ are of order 3.

Theorem 4. Let operators $L_1(\lambda)$ and $L_2(\lambda)$ be given by formulas (3–7). Then for the operator

$$\mathcal{L}(\lambda) = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{pmatrix} = L_1(\lambda) L_2(\lambda) L_2^\dagger(\lambda) L_1^\dagger(\lambda)$$

there exists, in the general position in the parameters of operators $L_1(\lambda)$ and $L_2(\lambda)$, the unique vector Ω from the auxiliary space such that $\mathcal{B}(\lambda) = 0$ for all λ . Vector Ω is an

eigenvector for operators $\mathcal{A}(\lambda)$ and $\mathcal{D}(\lambda)$ with eigenvalues $a(\lambda)$ and $d(\lambda)$, and the function $d(\lambda)$ (resp. $a(\lambda)$) has zeros of the first order in those and only those points where $\det L_1 \det L_2 = 0$ (resp. $\det \hat{L}_1 \det \hat{L}_2 = 0$). The vacuum curve of operator $\mathcal{L}(\lambda)$ has the equation $(v^3 - u^3)^{27} = 0$.

Proof. According to Lemma 3, the vacuum curve of operator $L_2(\lambda)L_2^\dagger(\lambda)$ has the form (11). It is easy to see that a curve of the form (11) commutes with any curve of the form (8). Thus, the vacuum curve of $\mathcal{L}(\lambda)$ coincides with the vacuum curve of $L_1(\lambda)L_1^\dagger(\lambda)L_2(\lambda)L_2^\dagger(\lambda)$, i.e. with the composition of two curves of the form (11). This composition yields the curve $(v^3 - u^3)^{27} = 0$.

It follows directly from the fact that the point $(u, v) = (0, 0)$ belongs to the vacuum curve of $\mathcal{L}(\lambda)$ for any λ that, for any λ , there exists such a vector $\Omega(\lambda)$ that $\mathcal{B}(\lambda)\Omega(\lambda) = 0$. It is known, however, that all operators $\mathcal{B}(\lambda)$ commute [1]. So, the space S^A where the operators $\mathcal{B}(\lambda)$ act decomposes into a direct sum of subspaces S_f^A corresponding to different “weights” $f(\lambda)$ in the sense that the operators $\mathcal{B}(\lambda) - f(\lambda) \cdot \mathbf{1}$ are nilpotent on S_f^A . It is clear from the above that the function $f(\lambda) \equiv 0$ is necessarily present among the “weights”. In the corresponding subspace S_0^A all the $\mathcal{B}(\lambda)$ are nilpotent and thus a vector $\Omega \in S_0^A$ exists such that $\mathcal{B}(\lambda)\Omega \equiv 0$.

Let us prove that the space of such vectors Ω is, in the general position, not more than one-dimensional. If $b_{13}^{(1)} = b_{13}^{(2)} = 0$ (formula (6)) and the other parameters are in the general position then necessarily

$$\Omega = \text{const} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The space of vectors Ω cannot have a smaller dimension in a particular case than in the general case.

The fact that Ω is an eigenvector for $\mathcal{A}(\lambda)$ and $\mathcal{D}(\lambda)$ follows now from the commutation relations of the 6-vertex model.

Clearly, the condition $\mathcal{D}(\lambda)\Omega = 0$ together with $\mathcal{B}(\lambda)\Omega = 0$ means the degeneracy of the operator $\mathcal{L}(\lambda)$; this can be achieved only in the points where $\det L_1 \cdot \det L_2 = 0$. Again from the case $b_{13}^{(1)} = b_{13}^{(2)} = 0$ we see that the function $d(\lambda)$ has zeros of the first order in all such points. This statement is extended to the general case by continuity, taking into account that the whole number of zeros of the function $d(\lambda)$ cannot change [4–7].

Similar arguments applied to the operator $\hat{\mathcal{L}}(\lambda) = \hat{L}_1^\dagger(\lambda)\hat{L}_2^\dagger(\lambda)\hat{L}_2(\lambda)\hat{L}_1(\lambda)$ gives the theorem’s statement as for function $a(\lambda)$. The theorem is proved.

Theorem 5. Let, in addition to the conditions of Theorem 4, the vacuum curves of operators $L_1(\lambda)$ and $L_2(\lambda)$ commute for all λ . Then the operator $\tilde{\mathcal{L}}(\lambda) = \begin{pmatrix} \tilde{\mathcal{A}}(\lambda) & \tilde{\mathcal{B}}(\lambda) \\ \tilde{\mathcal{C}}(\lambda) & \tilde{\mathcal{D}}(\lambda) \end{pmatrix} = L_2(\lambda)L_1(\lambda)L_2^\dagger(\lambda)L_1^\dagger(\lambda)$, too, possesses in the general position the unique vector $\tilde{\Omega}$ such that $\tilde{\mathcal{B}}(\lambda)\tilde{\Omega} = 0$, $\tilde{\mathcal{A}}(\lambda)\tilde{\Omega} = \tilde{a}(\lambda)\tilde{\Omega}$, $\tilde{\mathcal{D}}(\lambda)\tilde{\Omega} = \tilde{d}(\lambda)\tilde{\Omega}$, and the zeros of $\tilde{a}(\lambda)$ and $\tilde{d}(\lambda)$ coincide, respectively, with the zeros of $a(\lambda)$ and $d(\lambda)$. The vacuum curves of $\tilde{\mathcal{L}}(\lambda)$ and $\mathcal{L}(\lambda)$ coincide.

Proof. The vacuum curves of operators $L_1(\lambda)$ and $L_2(\lambda)$ commute, consequently the vacuum curve of $\tilde{\mathcal{L}}(\lambda)$ coincides with the vacuum curve of $\mathcal{L}(\lambda)$. The further reasonings are completely analogous to the proof of Theorem 4. The theorem is proved.

It is clear from formulas (3–7) that $a(\lambda)$, $d(\lambda)$, $\tilde{a}(\lambda)$ and $\tilde{d}(\lambda)$ must be trigonometric polynomials. As the zeros of the functions coincide as stated in Theorem 5, $\tilde{a}(\lambda) = a_0 a(\lambda)$ and $\tilde{d}(\lambda) = d_0 d(\lambda)$, where a_0 and d_0 are constants. It follows from this and from the theory of commutation relations of the 6-vertex model that the one-parametric family of operators $\tilde{\mathcal{L}}(\lambda)$ is *equivalent* to the family

$$\mathcal{M}(\lambda) = \begin{pmatrix} a_0 \mathcal{A}(\lambda) & \sqrt{a_0 d_0} \mathcal{B}(\lambda) \\ \sqrt{a_0 d_0} \mathcal{C}(\lambda) & d_0 \mathcal{D}(\lambda) \end{pmatrix}.$$

At the same time, the vacuum curve equation for the operator $\mathcal{L} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$ has the form [2]

$$\det \left[(1 \quad -v) \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \begin{pmatrix} u \\ 1 \end{pmatrix} \right] = 0 \quad (12)$$

(here the operator in square brackets acts in the auxiliary space). Thus, if the vacuum curve of operator $\mathcal{L}(\lambda)$ is given by the equation $\mathcal{P}(u, v) = 0$ then the vacuum curve of operator $\mathcal{M}(\lambda)$ must, as is easily seen, have the form $\mathcal{P}(\sqrt{\frac{a_0}{d_0}} u, \sqrt{\frac{d_0}{a_0}} v) = 0$. According to Theorems 4 and 5, both those curves have equations $(v^3 - u^3)^{27} = 0$. Hence, $a_0 = d_0$, and the operator family $\tilde{\mathcal{L}}(\lambda)$ is equivalent to the family $a_0 \mathcal{L}(\lambda)$. We see from the case $L_2(\lambda) = L_1(\lambda)$ that $a_0 \equiv 1$ (because a_0 as a function of the parameters entering in families $L_1(\lambda)$ and $L_2(\lambda)$ cannot have jumps). Thus, we proved the following

Theorem 6. If the vacuum curves of operators $L_1(\lambda)$ and $L_2(\lambda)$ given by formulas (3–7) commute for all λ then the family of operators $L_1(\lambda)L_2(\lambda)L_2^\dagger(\lambda)L_1^\dagger(\lambda)$ is equivalent to the family $L_2(\lambda)L_1(\lambda)L_2^\dagger(\lambda)L_1^\dagger(\lambda)$.

Remark. Similarly to Theorem 6, we can also show the family $L_2^\dagger(\lambda)L_1^\dagger(\lambda)L_1(\lambda)L_2(\lambda)$ to be equivalent to the family $L_2^\dagger(\lambda)L_1^\dagger(\lambda)L_2(\lambda)L_1(\lambda)$.

Consider now the following two operator families:

$$\mathcal{L}(\lambda)\mathcal{L}(\lambda) = L_1(\lambda)L_2(\lambda)L_2^\dagger(\lambda)L_1^\dagger(\lambda)L_1(\lambda)L_2(\lambda)L_2^\dagger(\lambda)L_1^\dagger(\lambda), \quad (13)$$

$$\mathcal{L}(\lambda)\tilde{\mathcal{L}}(\lambda) = L_1(\lambda)L_2(\lambda)L_2^\dagger(\lambda)L_1^\dagger(\lambda)L_2(\lambda)L_1(\lambda)L_2^\dagger(\lambda)L_1^\dagger(\lambda) \quad (14)$$

(recall that different copies of operators act in different auxiliary spaces). The families (13) and (14) have “generating vectors” $\Omega \otimes \Omega$ and $\Omega \otimes \tilde{\Omega}$ respectively, so it follows from the theory of 6-vertex model’s commutation relations that if these two families are equivalent then the operator R performing this equivalence is determined uniquely up to a numeric factor (in the general position). At the same time, we get from Theorem 6 and the following Remark *two* operators R acting in *different* auxiliary spaces (and multiplied by **1** in the rest, see Remark after Definition 2). Thus, in reality, R acts nontrivially only in the tensor product of the auxiliary spaces of operators $L_1(\lambda)$ and $L_2(\lambda)$ that are situated *on the right* in formulas (13) and (14); with this, Theorem 3 is proved.

§4. Discussion of results

Formulas (8) and (9) show that the operator R performing the equivalence

$$RL_1(\lambda)L_2(\lambda) = L_2(\lambda)L_1(\lambda)R \quad (15)$$

of operator families given by (3–7) exists if and only if the following equality holds for all λ :

$$\frac{a_1^3 \sin 3(\lambda + \rho_1) - d_1^3 \sin 3(\lambda + \sigma_1)}{b_{21}^{(1)} b_{32}^{(1)} b_{13}^{(1)}} = \frac{a_2^3 \sin 3(\lambda + \rho_2) - d_2^3 \sin 3(\lambda + \sigma_2)}{b_{21}^{(2)} b_{32}^{(2)} b_{13}^{(2)}}. \quad (15)$$

At the same time, if the equivalence (15) holds then vacuum curves of the operator families $L_1(\mu + \lambda)$ and $L_2(\lambda)$, $\mu = \text{const}$, will *no longer* commute (except for some special cases). Hence, due to Lemma 1, for such families the equivalence similar to (15) does not take place. This means that one cannot introduce a parameter μ similar to the parameter λ in $L_i(\lambda)$ in the set of all operators R performing equivalences of the type considered in this paper. The author hopes to investigate the issues of parameterization and the further properties of operators R in the future. Here let us mention the two following facts.

Theorem 7. Let there be *three* operator families $L_i(\lambda)$, $i = 1, 2, 3$, given by formulas (3–7) and having mutually commuting vacuum curves for all λ . Let operators R_{ij} , $i, j = 1, 2, 3$, $i < j$, perform equivalences $R_{ij}L_i(\lambda)L_j(\lambda) = L_j(\lambda)L_i(\lambda)R_{ij}$. Then the following equality (“triangle equation”) holds:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \quad (17)$$

Proof. The left- and right-hand sides of equation (17) both perform the equivalence between operator families $L_1(\lambda)L_2(\lambda)L_3(\lambda)$ and $L_3(\lambda)L_2(\lambda)L_1(\lambda)$. We can introduce three more operators and auxiliary spaces and say that both sides of (17) perform the equivalence between families $L_1^\dagger(\lambda)L_2^\dagger(\lambda)L_3^\dagger(\lambda)L_1(\lambda)L_2(\lambda)L_3(\lambda)$ and $L_1^\dagger(\lambda)L_2^\dagger(\lambda)L_3^\dagger(\lambda)L_3(\lambda)L_2(\lambda)L_1(\lambda)$. One can show in a way similar to the proofs of Theorems 4 and 5 that the two latter families possess “generating vectors” and thus, according to the theory of 6-vertex model’s commutation relations, the operator performing their equivalence is determined, in the general case, uniquely (to within a constant factor). The theorem is proved.

One more fact of interest is also that vacuum curves of the form (10) reveal obvious analogy to the vacuum curves of the well-known Felderhof model studied for the first time in paper [2].

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Remarks of year 2000

I hope that this paper is still of mathematical interest. On the historical side, in order to prove who was really first in discovering some facts related to the 6-vertex and chiral Potts models, I would like to add here some more references, namely to papers that I received back from journals in 1986–87. In particular, the same as in this paper was already done not only for 3, but for any number of “colors”. These are:

- I.G. Korepanov, Vacuum curves of \mathcal{L} -operators associated with the 6-vertex model, and construction of R -operators. Deposited at VINITI (“All-Union Institute for Scientific and Technical Information”) on April 2, 1986, Manuscript no. 2271-V86 (in Russian).
- I.G. Korepanov, Hidden symmetries of the 6-vertex model. Deposited at VINITI on February 27, 1987, Manuscript no. 1472-V87 (in Russian).
- I.G. Korepanov, On the spectrum of the transfer matrix of 6-vertex model. Deposited at VINITI on May 7, 1987, Manuscript no. 3268-V87 (in Russian).

Now this all (except some unimportant things) is published in the following papers:

- I.G. Korepanov, Vacuum curves of the \mathcal{L} -operators related to the six-vertex model, St. Petersburg. Math. J., V. 6, no. 2 (1995), 349–364.
- I.G. Korepanov, Hidden symmetries of the 6-vertex model of statistical physics, Zap. Nauch. Semin. POMI, V. 215 (1994), 163–177 (in Russian; English translation see in hep-th/9410066).